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On the Continuity of a Lebesgue Integral with Respect to a Parameter.

By J. K. Lamond.

Introduction.

Let $f(x, y)$ be a function of two variables defined over a limited, two-dimensional, measurable point set which may be denoted by $\mathfrak{A}$. Let $\mathfrak{B}$ be the projection of $\mathfrak{A}$ on the X-axis, and through each point x of $\mathfrak{B}$ let an ordinate be erected. Each ordinate cuts a section $\mathfrak{C}(x)$ out of $\mathfrak{A}$. When no confusion can arise, we denote this section simply by $\mathfrak{C}$. If each $\mathfrak{C}$ is measurable, and $f(x, y)$ is L-integrable in the sense of Professor Pierpont,[*] the resulting function

$$J(x) = \int_{\mathfrak{C}} f(x, y)\, dy$$

is a function of the parameter x. In the present paper I give sufficient conditions that

$$\lim_{x=\lambda} J(x) = \int_{\mathfrak{C}(\lambda)} f(\lambda, y)\, dy,$$

and consequently that $J(x)$ be a continuous function of x in $\mathfrak{B}$.

It is understood that the integral signs used denote L-integrals. In case a Riemann integral is used, the sign of integration will be prefixed by the letter R.

The upper measure of a point set $\mathfrak{A}$ will be denoted by the symbol $\overline{\text{meas.}}\, \mathfrak{A}$ or $\overline{\overline{\mathfrak{A}}}$. If $\mathfrak{A}$ is measurable, its measure will be denoted by meas. $\mathfrak{A}$ or $\hat{\hat{\mathfrak{A}}}$.[†]

§ 1. *Proper Integrals.*

Theorem 1.[‡] *Let $0 \leq f \leq N$ be L-integrable in measurable $\mathfrak{A}$, and let $\mathfrak{A}_\kappa$ be the points of $\mathfrak{A}$ at which $f \geq \kappa$. Then*

$$\int_{\mathfrak{A}} f = R \int_0^N \hat{\hat{\mathfrak{A}}}_\kappa\, d\kappa. \tag{1}$$

Effect a division of the interval $(0, N)$ of norm d by interpolating the points $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$. For uniformity let $\kappa_0 = 0$ and $\kappa_n = N$. Corresponding to these points, we get the sets $\mathfrak{A}_{\kappa_0}, \mathfrak{A}_{\kappa_1}, \ldots, \mathfrak{A}_{\kappa_n}$ each of which is measurable, by "Lectures," Vol. II, §§ 424 and 408, 2, and each of which contains the following ones. The sets $(\mathfrak{A}_{\kappa_0}-\mathfrak{A}_{\kappa_1}), (\mathfrak{A}_{\kappa_1}-\mathfrak{A}_{\kappa_2}), \ldots, (\mathfrak{A}_{\kappa_{n-1}}-\mathfrak{A}_{\kappa_n})$ are all measurable; and meas. $(\mathfrak{A}_{\kappa_\iota}-\mathfrak{A}_{\kappa_{\iota+1}}) = \hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_\iota}-\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_{\iota+1}}$, $(\iota = 0, 1, 2, \ldots, n-1)$, by "Lectures," Vol. II, § 352, 2.

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[*] "Lectures on the Theory of Functions of Real Variables," Vol. II, § 380. These will be referred to as "Lectures."

[†] This is the notation adopted in "Lectures," Vol. II.

[‡] This theorem is similar to one given by W. H. Young in a paper "On the General Theory of Integration," *Phil. Trans.*, Vol. CCIV (1905).

But obviously any lower summation $\le$ the L-integral of f over $\mathfrak{A} \le$ any upper summation. Hence,

$$0 \text{ meas. }(\mathfrak{A}_{\kappa_0}-\mathfrak{A}_{\kappa_1}) + \kappa_1 \cdot \text{meas. }(\mathfrak{A}_{\kappa_1}-\mathfrak{A}_{\kappa_2}) + \ldots + \kappa_{n-1} \cdot \text{meas. }(\mathfrak{A}_{\kappa_{n-1}}-\mathfrak{A}_{\kappa_n})$$

$$\le \int_{\mathfrak{A}} f \le$$

$$\kappa_1 \cdot \text{meas. }(\mathfrak{A}_{\kappa_0}-\mathfrak{A}_{\kappa_1}) + \kappa_2 \cdot \text{meas. }(\mathfrak{A}_{\kappa_1}-\mathfrak{A}_{\kappa_2}) + \ldots + \kappa_n \cdot \text{meas. }(\mathfrak{A}_{\kappa_{n-1}}-\mathfrak{A}_{\kappa_n}).$$

Hence, removing the parentheses, we have

$$(\kappa_1 \underset{-\iota_0)}{\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_1}} + (\kappa_2-\kappa_1)\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_2} + \ldots + (\kappa_{n-1}-\kappa_{n-2})\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_{n-1}} - \kappa_{n-1}\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_n}$$

$$\le \int_{\mathfrak{A}} f \le$$

$$\kappa_1 \hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_0} + (\kappa_2-\kappa_1)\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_1} + \ldots + (\kappa_n-\kappa_{n-1})\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_{n-1}} - \kappa_n\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_n}.$$

Noticing that $\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_n} = 0$, this may be written

$$\sum_0^{n-2}(\kappa_{\iota+1}-\kappa_\iota)\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_{\iota+1}} \le \int_{\mathfrak{A}} f \le \sum_0^{n-1}(\kappa_{\iota+1}-\kappa_\iota)\hat{\hat{\hat{\mathfrak{A}}}}_{\kappa_\iota}.$$

Let now $d = 0$, and we have

$$R\,\underline{\int}_0^N \hat{\hat{\hat{\mathfrak{A}}}}_\kappa\, d\kappa \le \int_{\mathfrak{A}} f \le R\,\overline{\int}_0^N \hat{\hat{\hat{\mathfrak{A}}}}_\kappa\, d\kappa. \tag{2}$$

But since $\hat{\hat{\hat{\mathfrak{A}}}}_\kappa$ is a limited, monotone function of κ in $(0, N)$, it is integrable. Hence, (2) gives (1).

COROLLARY 1. *Let $M < f \le 0$ be L-integrable in measurable $\mathfrak{A}$. Let $\mathfrak{A}'_\kappa$ be the points of $\mathfrak{A}$ at which $f \le \kappa$. Then*

$$\int_{\mathfrak{A}} f = R\int_M^0 \hat{\hat{\hat{\mathfrak{A}}}}'_\kappa\, d\kappa.$$

COROLLARY 2. *Let f be an L-integrable function of both signs in measurable $\mathfrak{A}$, such that $M < f < N$. Let $\mathfrak{A}_\kappa$ be the points of $\mathfrak{A}$ at which $f \ge \kappa$, and $\mathfrak{A}'_\kappa$ be the points of $\mathfrak{A}$ at which $f \le \kappa$. Then*

$$\int_{\mathfrak{A}} f = R\int_M^0 \hat{\hat{\hat{\mathfrak{A}}}}'_\kappa\, d\kappa + R\int_0^N \hat{\hat{\hat{\mathfrak{A}}}}_\kappa\, d\kappa, \tag{1}$$

$$= -\int_M^0 \hat{\hat{\hat{\mathfrak{A}}}}'_\kappa\, d\kappa + \int_0^N \hat{\hat{\hat{\mathfrak{A}}}}_\kappa\, d\kappa. \tag{2}$$

Let the points of $\mathfrak{A}$ for which $M < f \le 0$ be A, for which $0 \le f < N$ be B, and for which $f = 0$ be C. By "Lectures," Vol. II, §§ 428 and 408, 1 and 2, A, B and C are all measurable. Hence, $A - C$ is measurable. $\mathfrak{A} = (A - C) + B$ and meas. $\mathfrak{A}$ = meas. $(A - C)$ + meas. B. Therefore, by "Lectures," Vol. II, §§ 372 and 390, 2,

$$\int_{\mathfrak{A}} f = \int_{A-C} f + \int_B f.$$

But by "Lectures," Vol. II, § 401,

$$\int_{A-C} f = \int_A f.$$

These relations, with the results of theorem 1 and corollary 1, give (1). We get (2) from (1) by referring to "Lectures," Vol. II, § 381, 2.

THEOREM 2. *Let λ be a proper limiting point of $\mathfrak{B}$. Let $M < f < N$ be L-integrable in each $\mathfrak{C}$, $\mathfrak{C}$ measurable, in some $V_\delta(\lambda)$,* and let $g(y)$ be L-integrable in measurable $\mathfrak{C}(\lambda)$. Let the points of $\mathfrak{C}(\lambda)$ at which $g(y) \leq \varkappa$ be ϕ'_κ, and at which $g(y) \geq \varkappa$ be ϕ_κ. Let $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ converge uniformly† to $\widehat{\widehat{\phi}}'_\kappa$ in $(M, 0)$, and let $\widehat{\widehat{\mathfrak{C}}}_\kappa$ converge uniformly to $\widehat{\widehat{\phi}}_\kappa$ in $(0, N)$, along the section $x = \lambda$, except possibly for the points of a null set $\mathfrak{N}$ of values of $\varkappa$. Then*

$$\lim_{x=\lambda} \int_{\mathfrak{C}(x)} f\, dy = \int_{\mathfrak{C}(\lambda)} g(y)\, dy. \tag{1}$$

For by theorem 1, corollary 2,

$$\int_{\mathfrak{C}} f\, dy = \int_M^0 \widehat{\widehat{\mathfrak{C}}}'_\kappa\, d\varkappa + \int_0^N \widehat{\widehat{\mathfrak{C}}}_\kappa\, d\varkappa,$$

for x in $V_\delta(\lambda)$. Pass to the limit $x = \lambda$, and we have

$$\lim_{x=\lambda} \int_{\mathfrak{C}(x)} f\, dy = \lim_{x=\lambda} \left[\int_M^0 \widehat{\widehat{\mathfrak{C}}}'_\kappa\, d\varkappa + \int_0^N \widehat{\widehat{\mathfrak{C}}}_\kappa\, d\varkappa\right]. \tag{2}$$

Let

$$D = \int_0^N \widehat{\widehat{\mathfrak{C}}}_\kappa\, d\varkappa - \int_0^N \widehat{\widehat{\phi}}_\kappa\, d\varkappa,$$

for x in $V_\delta(\lambda)$. For any x in $V_\delta(\lambda)$ and for any $\varkappa$, $\widehat{\widehat{\mathfrak{C}}}_\kappa$ and $\widehat{\widehat{\phi}}_\kappa$ are both less than or equal to $\bar{\bar{\Delta}}$, where Δ is the projection of $\mathfrak{A}$ on the Y-axis. Choosing $\varepsilon > 0$, small at pleasure, we may enclose the points of $(0, N)$ belonging to $\mathfrak{N}$ in a set of non-overlapping intervals B, such that $\bar{\bar{B}} < \frac{\varepsilon}{4\bar{\bar{\Delta}}}$. Let the remaining

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* $V\delta(\lambda)$ denotes those points of $\mathfrak{B}$ lying in the interval $\left(\lambda - \frac{\delta}{2}, \lambda + \frac{\delta}{0}\right)$, and is read *vicinity of* λ, *of norm* δ. This is the notation used in "Lectures." (See Vol. I, § 250.)

† See "Lectures," Vol. I, § 561. In this particular case, if, for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$| \widehat{\widehat{\mathfrak{C}}}'_\kappa - \widehat{\widehat{\phi}}'_\kappa | < \epsilon \tag{1}$$

for each x in $V\delta(\lambda)$, and for every κ in $(M, 0)$, we say that $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ converges to $\widehat{\widehat{\phi}}'_\kappa$ *uniformly along the section* $x = \lambda$.

If $\mathfrak{N}$ is a null set of values of κ lying in $(M, 0)$, there exists a division of $(M, 0)$ into two sets of intervals, A and B, such that A contains no points of $\mathfrak{N}$ and the sum of the lengths of the intervals B is as small as we please. If for each $\epsilon > 0$, and for any A, there exists a $\delta > 0$, such that (1) holds for each x in $V\delta(\lambda)$, and for every κ in A, we say that $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ converges to $\widehat{\widehat{\phi}}'_\kappa$ *uniformly along the section* $x = \lambda$ *except for the points of* $\mathfrak{N}$. If $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ converges to $\widehat{\widehat{\mathfrak{C}}}'_\kappa(\lambda)$ uniformly along the section $x = \lambda$, except for the points of $\mathfrak{N}$, we say that $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ is a *uniformly continuous function of* x *along the section* $x = \lambda$ *except for the points of* $\mathfrak{N}$. If $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ is a uniformly continuous function of x along each section $x = \lambda$ of $\mathfrak{B}$, except for the points of $\mathfrak{N}$, we say that $\widehat{\widehat{\mathfrak{C}}}'_\kappa$ is a *uniformly continuous function of* x *in* $\mathfrak{B}$ *except for the points of* $\mathfrak{N}$.

parts of $(0, N)$ be A. Then

$$D = \int_A [\hat{\hat{\mathfrak{E}}}_\kappa - \hat{\hat{\phi}}_\kappa]\, d\varkappa + \int_B [\hat{\hat{\mathfrak{E}}}_\kappa - \hat{\hat{\phi}}_\kappa]\, d\varkappa.$$

Hence,

$$| D | \leq | \int_A | + | \int_B |. \tag{3}$$

But the second part of the right-hand member of (3) is less than $\frac{\varepsilon}{2}$. On the other hand, by hypothesis,

$$| \hat{\hat{\mathfrak{E}}}_\kappa - \hat{\hat{\phi}}_\kappa | < \frac{\varepsilon}{2N},$$

for x in $V_\delta(\lambda)$ and $\varkappa$ in A. Hence, the first part of the right-hand member of (3) is less than $\frac{\varepsilon}{2}$. Hence, $| D | < \varepsilon$, for x in $V_\delta(\lambda)$. Similarly,

$$| D' | = | \int_M^0 \hat{\hat{\mathfrak{E}}}'_\kappa\, d\varkappa - \int_M^0 \hat{\hat{\phi}}'_\kappa\, d\varkappa | < \varepsilon,$$

for x in $V_\delta(\lambda)$. These results, with (2), give

$$\lim_{x=\lambda} \int_{\mathfrak{E}(x)} f\, dy = \int_M^0 \hat{\hat{\phi}}'_\kappa\, d\varkappa + \int_0^N \hat{\hat{\phi}}_\kappa\, d\varkappa = \int_{\mathfrak{E}(\lambda)} g(y)\, dy,$$

by theorem 1.

COROLLARY 1. *Let f be L-integrable in each $\mathfrak{E}$, $\mathfrak{E}$ measurable, in $\mathfrak{B}$. Let $\hat{\hat{\mathfrak{E}}}'_\kappa$ and $\hat{\hat{\mathfrak{E}}}_\kappa$ be uniformly continuous functions of x in $\mathfrak{B}$, except possibly for a null set $\mathfrak{N}$ of values of x. Then*

$$J(x) = \int_{\mathfrak{E}} f\, dy$$

is continuous in $\mathfrak{B}$.

Example 1. Let f be defined over a set $\mathfrak{A}$, which lies in the rectangle 0, 0; 1, 0; 0, 2; 1, 2, as follows:

$$\begin{aligned} f &= \frac{1}{nq} \text{ for } x = \frac{m}{n},\ y = \frac{p}{q},^{*} \\ &= xy \text{ for } x = \frac{m}{n},\ 0 \leq y \neq \frac{p}{q} \leq 1, \\ &= x\,[y - 1) \text{ for } x \text{ irrational},\ 1 \leq y \neq \frac{p}{q} \leq 2. \end{aligned}$$

Setting $\varkappa = xy$ and solving for y, we have $y = \frac{\varkappa}{x}$. Therefore, $\hat{\hat{\mathfrak{E}}}_\kappa$ equals $1 - \frac{\varkappa}{x}$, for $\varkappa \leq x$, and equals zero for $\varkappa > x$, where x is rational; that is, of the form $\frac{m}{n}$. Setting $\varkappa = x\,(y - 1)$ and solving for y, we have $y = \frac{\varkappa}{x} - 1$. Hence, $\hat{\hat{\mathfrak{E}}}_\kappa$

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[*] All fractions, here and elsewhere, are supposed to be irreducible.

equals $2 - \frac{\kappa}{x} - 1 = 1 - \frac{\kappa}{x}$, for $\kappa \leq x$, and equals zero for $\kappa > x$, where x is irrational. Thus $\hat{\hat{\mathfrak{E}}}_\kappa$ is a uniformly continuous function of x for any κ, and hence, $J(x)$ is continuous in $\mathfrak{B} = (0, 1)$.

Example 2. Let f be defined as in example 1 for x rational. For x irrational, and $\frac{x}{2} \leq y \leq \frac{x}{2} + 1$, let $f = xy - \frac{x^2}{2}$. Let $\kappa = xy - \frac{x^2}{2}$. Then $y = \frac{\kappa}{x} + \frac{x}{2}$, and $\hat{\hat{\mathfrak{E}}}_\kappa$ equals $\frac{x}{2} + 1 - \left(\frac{\kappa}{x} + \frac{x}{2}\right) = 1 - \frac{\kappa}{x}$, for $\kappa \leq x$, and equals zero for $\kappa > x$. Hence, $J(x)$ is continuous in $\mathfrak{B}$, as in example 1.

§ 2. *Improper Integrals.*

THEOREM 3. *Let λ be a proper limiting point of $\mathfrak{B}$. Let the improper L-integral* of f over each $\mathfrak{E}$, $\mathfrak{E}$ measurable, exist and converge uniformly† in some $V_\delta(\lambda)$, and let $g(y)$ be improperly L-integrable in measurable $\mathfrak{E}(\lambda)$. For any arbitrary but fixed values of α, β, such as a, b, let the points of $\mathfrak{E}_{ab}(\lambda)$ at which $g(y) \leq \kappa$ be $\phi'_{ab,\kappa}$, and at which $g(y) \geq \kappa$ be $\phi_{ab,\kappa}$. Let $\hat{\hat{\mathfrak{E}}}_{ab,\kappa}$ converge uniformly to $\hat{\hat{\phi}}_{ab,\kappa}$ in $(-a, 0)$, and let $\hat{\hat{\mathfrak{E}}}_{ab,\kappa}$ converge uniformly to $\hat{\hat{\phi}}_{ab,\kappa}$ in $(0, b)$, along the section $x = \lambda$, except possibly at the points of a null set $\mathfrak{N}$ of values of κ. Then*

$$\lim_{x=\lambda} \int_{\mathfrak{E}(x)} f\,dy = \int_{\mathfrak{E}(\lambda)} g(y)\,dy.$$

For any x in $V_\delta(\lambda)$, $\mathfrak{E}_{ab}$ is measurable. Hence, by theorem 2,

$$\lim_{x=\lambda} \int_{\mathfrak{E}_{ab}(x)} f\,dy = \int_{\mathfrak{E}_{ab}(\lambda)} g(y)\,dy. \qquad (2)$$

By "Lectures," Vol. II, § 146, passing to the limit $a, b = \infty$ in (2), we get (1).

COROLLARY 1. *Let the improper L-integral of f over measurable $\mathfrak{E}$ converge uniformly in $\mathfrak{B}$. For any arbitrary but fixed values of α, β, such as a, b, let $\hat{\hat{\mathfrak{E}}}'_{ab,\kappa}$ and $\hat{\hat{\mathfrak{E}}}_{ab,\kappa}$ be uniformly continuous functions of x in $\mathfrak{B}$, except possibly for a null set $\mathfrak{N}$ of values of κ. Then*

$$J(x) = \int_{\mathfrak{E}} f\,dy$$

is continuous in $\mathfrak{B}$.

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* The definition of an improper L-integral given by Professor Pierpont in "Lectures," Vol. II, § 425, is here used. The modification of the notation there employed to meet our present needs seems obvious.

† The definition of uniform convergence for Pierpont improper integrals as given by me in my paper "Improper Multiple Integrals over Iterable Fields," *Transactions of the American Mathematical Society*, Vol. XIII (1912), p. 436, applies equally well to improper L-integrals.